Stochastic Modeling of Multiphase Flow in Porous Media

M. Tyagi, P. Jenny, H. A. Tchelepi

Abstract

A stochastic framework for modeling multiphase flow in porous media at the Darcy scale is presented. The basic idea is to move the stochastic particles with phase velocities that are consistent with the small scale physics. These particles serve as a realization for the flow variables, and thus, one has the whole statistics (joint PDF) of the flow. Such a framework can be used for the compositional modeling, where complex nonlinear processes take place at a small scale. As an illustrative example we consider viscous fingering in the displacement of two immiscible liquids. We show how one can construct a stochastic system to obtain the flow statistics as one obtains from the direction numerical simulation .

Joint PDF Transport Equation

Let us consider an incompressible system of n-immiscible phases flowing in a porous media. We are interested in knowing the joint statistics of the flow velocities. Let $\hat{A}$ be a random variable which can take only positive integral values where each value identifies a phase. Let f be the joint probability density function of $\hat{V}_1^{\hat{A}},\hat{V}_2^{\hat{A}},\hat{V}_3^{\hat{A}},\hat{A}$ in the velocity and phase space (u_1,u_2,u_3,a) . Its evolution is given by the following high dimensional partial differential equation.

$$\frac{\partial f}{\partial t}+\frac{\partial}{\partial x_j}\left(\left\langle \left.\frac{d\hat{X}_j^{\hat{A}}}{dt}\right|_{\underline{u},a}\right\rangle f\right)+\frac{\partial}{\partial u_j}\left(\left\langle \left.\frac{d\hat{V}_j^{\hat{A}}}{dt}\right|_{\underline{u},a}\right\rangle f\right)+\frac{\partial}{\partial a}\left(\left\langle \left.\frac{d\hat{A}}{dt}\right|_{\underline{u},a}\right\rangle f\right)=0 \qquad (1)$$

All expectations $\langle \ldots|_{\underline{u},a}\rangle$ in the above equation are conditioned on the velocity-phase space. While $\left\langle d\hat{A}/dt|_{u,\underline{a}}\right\rangle=0$ and $\left\langle d\hat{X}_j^{\hat{A}}/dt|_{u,\underline{a}}\right\rangle=u_j$, are closed, $\left\langle d\hat{V}_j^{\hat{A}}/dt|_{u,\underline{a}}\right\rangle$ has to be modeled.

Averaged Equations

Integration of Eq. (1) over the velocity space conditioned on $a=\alpha$ yields the following transport equation for the saturation of phase α .

$$\frac{\partial S^\alpha}{\partial t}+\frac{\partial}{\partial x_j}\left(\left\langle \hat{V}_j^{\hat{A}}\Big|_{a=\alpha}\right\rangle S^\alpha\right)=0 \qquad (2)$$

For closure, only the average phase particle velocity has to be known. Therefore, if one can model average particle velocities of a phase, the system of equations for saturation transport is closed. Note that the condition

$$\sum_{\alpha=1}^{n}\frac{\partial}{\partial x_j}\left(\left\langle \hat{V}_j^{\hat{A}}\Big|_{a=\alpha}\right\rangle S^\alpha\right)=0 \qquad (3)$$

must be fulfilled. Depending on the scenario, the averaged equations for the higher moments are not closed. For example, multiplying Eq. (1) by u_k and integrating over whole velocity space conditioned on $a=\alpha$ yields

$$\frac{\partial}{\partial t}\left(\left\langle \hat{V}_k^{\hat{A}}\Big|_{a=\alpha}\right\rangle S^\alpha\right)+\frac{\partial}{\partial x_j}\left(\left\langle \hat{V}_k^{\hat{A}}\hat{V}_j^{\hat{A}}\Big|_{a=\alpha}\right\rangle S^\alpha\right)+S^\alpha\left\langle \left.\frac{d\hat{V}_j^{\hat{A}}}{dt}\right|_{a=\alpha}\right\rangle=0 \qquad (4)$$

Above equation shows how the average particle velocities depend on their higher moments. While both, the second and third terms in Eq. (4) appear unclosed, only the conditional acceleration $\left\langle d\hat{V}_j^{\hat{A}}/dt|_{u,\underline{a}}\right\rangle$ in Eq. (1) has to be modeled, which in general is a much easier task.

Stochastic Particle Method for Solving Joint PDF Equation

Eq. (1) is a six dimensional equation and solving such a equation with finite volume method could be very expensive. An alternative way is to perform Monte Carlo simulation for solving the same problem. Such a method has recently been developed by *Tyagi et al.* where the phase particles are transported according to the following stochastic differential equations

$$\frac{dX_j^*}{dt}=V_j^*, \qquad \frac{dV_j^*}{dt}=\ldots \qquad (5)$$

where x_j^* and v_j^* are the particle position and velocity, respectively.

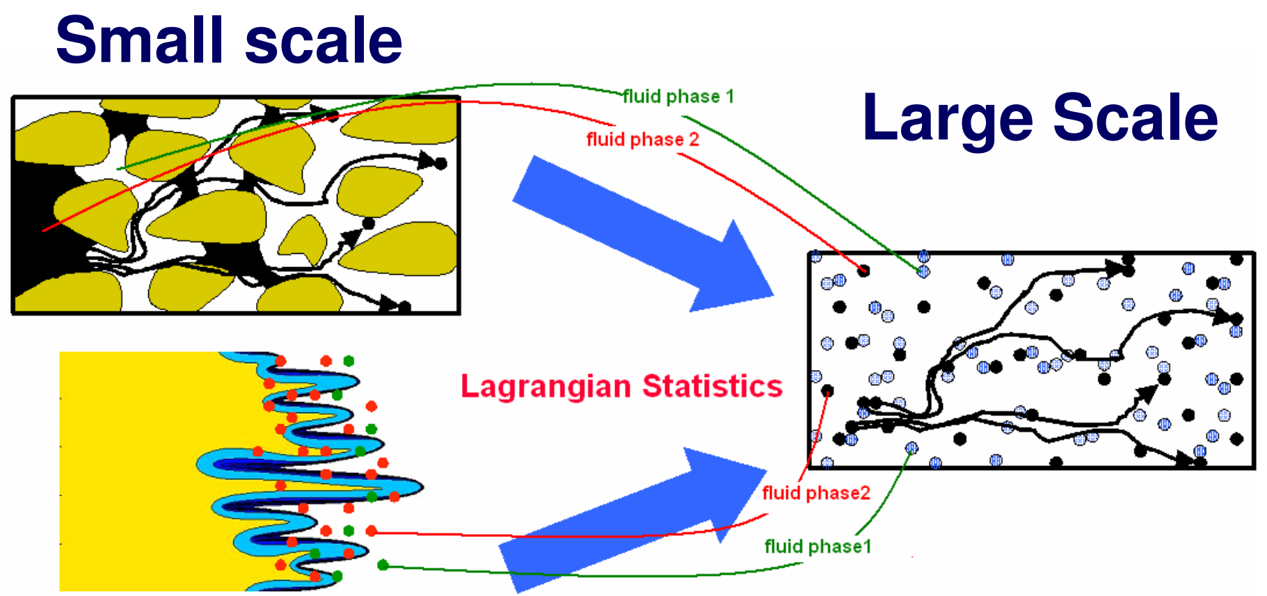

Fig.1 The sketch shows how phase particles can be used to mimic the small scale dynamics at the large scale.

The approach is very general: $\left\langle d\hat{V}_j^{\hat{A}}/dt|_{u,\underline{a}}\right\rangle$ may be described by some stochastic model, which mimics the physics at the small, unresolved scale relevant for the scenario considered. The statistics from the ensemble of particles then reflects the averaged macroscopic behavior.

An Illustrative Example: Stochastic Modeling of Viscous Fingering

Fig. 2 shows a result obtained with direct numerical simulation of a two dimensional immiscible displacement problem where a highly viscous fluid is displaced by a less viscous fluid (viscosity ratio = 20). Viscous instability leads to fingering, which makes the fluid distribution heterogeneous near the saturation front.

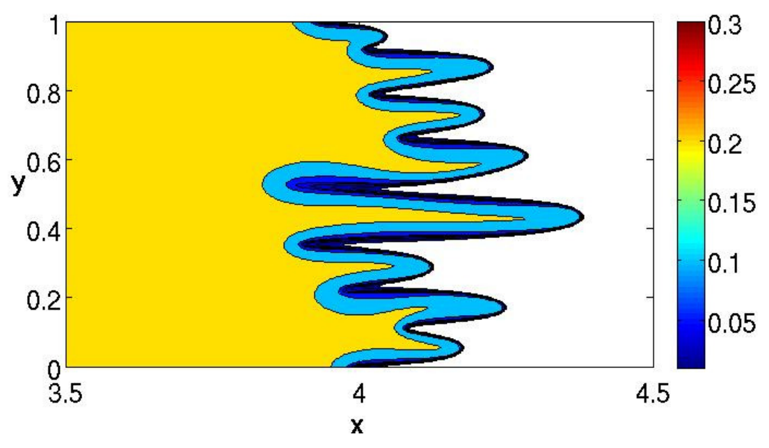

Fig. 2 Saturation contours of invading phase

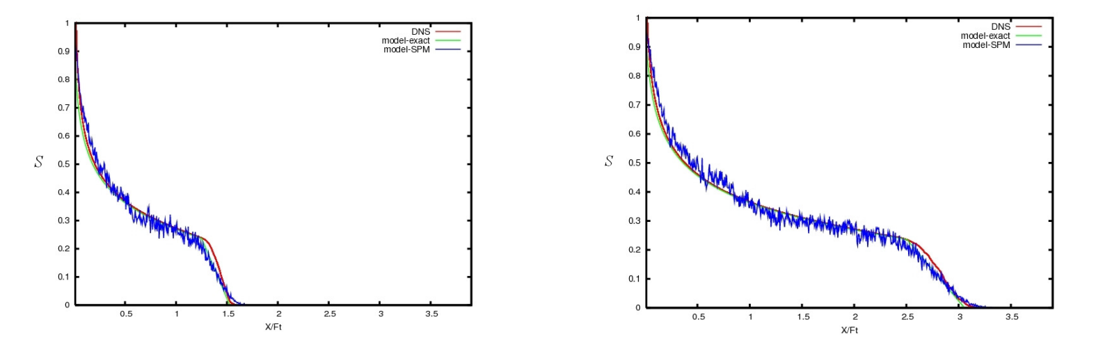

Fig. 3 Ensemble (or laterally) averaged saturation at two different times

Consider the transverse flow distribution at some axial location in the fingering regime which represents the statistics at that position. We want to construct a one dimensional stochastic model which leads to the same flow distribution.

A model for the average particle velocity

As a starting point, we consider a simplified model by specifying the average particle velocity. The idea is to get the same evolution of the average saturation as in the DNS (red curve in Fig. 3). In the absence of fingering (stable front), the saturation profile has a shock followed by an expansion fan (Fig. 4). The point of intersection of shock and expansion fan has a fixed saturation value (S^t) . In the case of fingering, the shock front is disturbed by a transverse random wave which grows in the size with time. If one neglects the dispersive effects (due to capillary forces), the fingers may be considered as two dimensional structures having a sharp curved front as sketched in Fig. 5. The saturation at the root of the finger is $S^r > S^t$. All invading phase particles in the fingering regime remain in the expansion fan. Therefore, their velocities do not vary much. Let us assume that invading phase particles undergo linear variation of velocity from the root to the tip of the fingers.

$$\langle V^1\rangle=\left(a\frac{S}{S^r}+b\right)F, \quad a=\frac{1-\beta}{2}ff'(S^r), \quad b=\beta ff'(S^r), \quad \beta=\frac{ff'(0)}{ff'(S^r)} \qquad (6)$$

where β is the finger growth rate factor and F is the average total flux. This velocity variation is valid only in the fingering regime. Now we find the average velocity of the other phase such that the following local mass balance is satisfied

$$\langle V^1\rangle S+\langle V^2\rangle(1-S)=\langle F\rangle \quad \Rightarrow \langle V^2\rangle=\frac{\langle F\rangle-\langle V^1\rangle S}{1-S} \qquad (7)$$

The green (exact solution) and blue (particle solution) curves in Fig. 3 are the saturation profiles obtained from the above model for β equal to 1.22.

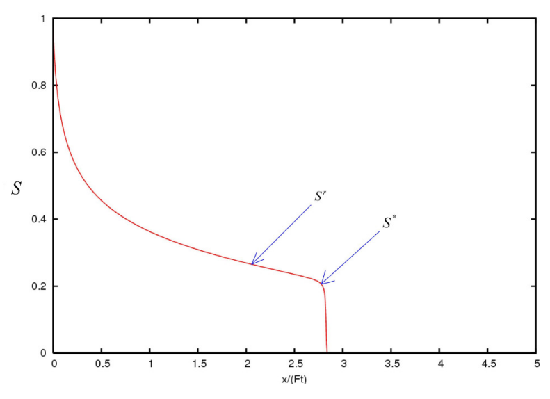

Fig. 4

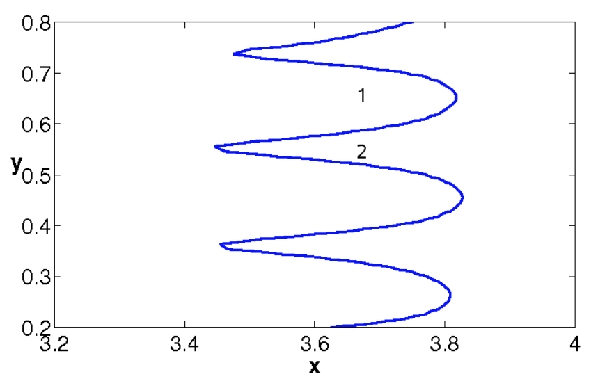

Fig. 5

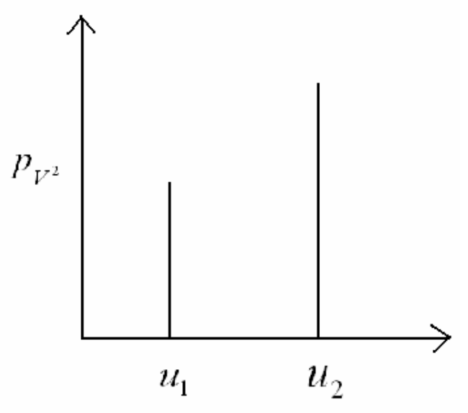

Fig.6 Bimodal velocity PDF for the displaced phase

A model for the particle velocity distribution

In the absence of dispersive effects, the velocity of invading phase particles does not experience much variation in the transverse direction. However, the velocity of particles representing displaced phase changes drastically as the particle crosses the shock. Consider two points 1 and 2 as shown in the Fig. 5. Assuming that the axial pressure gradient is constant along the transverse direction the particle velocity ratio is (for quadratic relative permeability)

$$u_1/u_2=\left(1-\hat{S}_1\right)/\left(1-\hat{S}_2\right)=\left(1-\hat{S}_1\right) \qquad (8)$$

where ^ denotes local quantities. The particle velocity PDF is bimodal with two peaks at u_1 and u_2

$$p_{V^2}(u)=(1-S)\delta(u-u_1)+S\delta(u-u_2) \qquad (9)$$

Using Eq. (7) and Eq. (8) one obtains the following values of u_1 and u_2

$$u_1=F/\left((1-S)\left(1-\hat{S}_1\right)+S\right) \qquad u_2=\left(1-\hat{S}_1\right)F/\left((1-S)\left(1-\hat{S}_1\right)+S\right) \qquad (10)$$

Conclusion

For this example, the PDF equation was only used to derive the averaged equation, which can be solved based on the proposed closure assumptions. More complex scenarios, however, often involve memory effects, which have to be considered for the particle evolution. In such cases, the statistics of the PDF equation using the stochastic particle method leads to a higher level of closure.